THAT STRIKES A CHORD! AN ILLUSTRATION OF PERMUTATION GROUPS IN MUSIC THEORY

ABSTRACT. We give several examples illustrating the prevalence of permutation groups in music theory, in particular using inversions and certain Music Theoretic functions called P, L, R functions, which arise in an area of study called Neo-Riemannian Theory. We find natural identifications of groups arising in this theory with well-known groups and continue an examination of the subgroup structure of the $\langle P, L, R\rangle$ group, expanding upon previous work in the literature. We then relate the T/I group, of musical Transposition and Inversion permutations, to a musical object called the Circle of Fifths. It is shown that a symbolic representation of the Circle of Fifths can be obtained from the orbit of a C_{major} under the group $\langle \mathcal{T}_7\rangle$.

1. INTRODUCTION

The use of major and minor triples - three notes played in tandem with a highly specific structure - in music is as prolific as it is essential and has a vast amount of hidden symmetry.

Group theory is a branch of the mathematical subject of abstract algebra that precisely describes the symmetries of diverse objects, which, as we will see herein, includes musical chords.

In this article, we provide a gentle introduction to the methods of this theory and some ways it can be used to describe the musical structure of chords and understand how they operate from a mathematical perspective. In particular, we examine major and minor triples to derive these mathematical relationships via the actions of mathematical objects called permutation groups.

Major and minor chords are named after their root, that is, the first note of the major or minor scale in which they belong, and consist of the root, the third, and the fifth of the corresponding scale. For musical purposes, the order of these notes does not change their identification. For example, a "c minor" chord consists of the notes $(c, e\flat, g)$ played simultaneously. While the inverted chord $(e\flat, g, c)$ has a different order, it is still considered a c minor chord.

In this article, we will examine three types of permutations that act upon such chords: chord inversions, which change the order of the notes; the P, L, R permutations of Neo-Riemannian theory, which map chords to related sets of tones; and transpositions, which shift notes within a set we identify with the symbol $\mathbb{Z}_{12}$.

Our explicit goal is that these discussions will illuminate hidden mathematical structure within the study of chords in a self-contained manner accessible to the curious undergraduate and professional mathematician alike. Through this article, we hope to introduce the reader to some of the ideas in the existing literature as well as introduce some new outlooks on the topic. We remark that in addition to the references cited in the course of this article, the book [1] by the well-known representation theorist D. Benson provides a very nice, detailed account of the various roles that mathematics plays in music, with a chapter dedicated to group theoretic applications. Readers interested in further reading are encouraged to explore this and the other references.

2. Background: Musical Chords

Here we will cover the bare essentials of music theory pertaining to our study of chords and introduce some mathematical interpretations widely used in the literature.

Musical notes are physical frequencies of sound waves that travel through the air and to the human ear. There are twelve distinct frequencies, called *semitones*, that are considered musically different. If the reader would imagine a piano keyboard, they will notice that there are repeating patterns of keys, each one corresponding directly to a semitone. Musically, we tend to begin at a semitone called "c". The keyboard ascends in pitch to the right with the following sequence of semitones beginning at c:

$$c, \quad c\sharp/d\flat, \quad d, \quad d\sharp/e\flat, \quad e, \quad f, \quad f\sharp/g\flat, \quad g, \quad g\sharp/a\flat, \quad a, \quad a\sharp/b\flat, \quad b, \quad c.$$

We notice that there are notes that have either a $\flat$ or $\sharp$ symbol on the right. These mean down one semitone and up one semitone, respectively. The notes such as $f\sharp/g\flat$ are called *enharmonics* and are musically equivalent to each other. The reader may think about enharmonic equivalence as one black key on a keyboard going by two names which are more indicative of location than anything else.

One full sequence through all uniquely identified semitones is called an *octave*. All the frequencies in a higher octave, (e.g. on the far right of a piano keyboard) are integer multiples of the frequencies in a lower octave (e.g. the far left on a piano keyboard). Because of this and because of enharmonic equivalence, it is mathematically much easier to view semitones as the integers modulo 12, written $\mathbb{Z}_{12}$. We may assign one value in $\mathbb{Z}_{12}$ to a semitone. Let the note in the table correspond to the number in $\mathbb{Z}_{12}$ as below:

$\mathbb{Z}_{12}$	0	1	2	3	4	5	6	7	8	9	10	11
$Note$	c	$c\sharp$	d	$d\sharp$	e	f	$f\sharp$	g	$g\sharp$	a	$a\sharp$	b

From this numerical interpretation, we may begin to examine the mathematical properties of musical chords.

Technically speaking, a musical chord really is any collection of the notes of $\mathbb{Z}_{12}$ played simultaneously. In particular, a chord may have any number of notes. For example, a trivial chord would be a single note. For our purposes, however, we will restrict our view to special chords called *major* or *minor triples*. Mathematically, these chords have a very specific structure, and, from the nomenclature, exactly three notes.

Take $x \in \mathbb{Z}_{12}$ to be the first note of a major triple. The major triple chord has structure as follows: $(x, x+4, x+7)$; and the minor triple has structure: $(x, x+3, x+7)$; where the addition is taken modulo 12. These objects are nearly ubiquitous in musical literature. There are exactly twenty-four major and minor triples, each with a different starting note from $\mathbb{Z}_{12}$, and they are named by this starting note, called the *root* of the chord. It is common musically to denote a major triple with the capital letter of the root and a minor triple with the lower case letter. We list these chords here:

$MajorRoot$	C	$C\sharp$	D	$D\sharp$	E	F
$Chord$	(0,4,7)	(1,5,8)	(2,6,9)	(3,7,10)	(4,8,11)	(5,9,0)
$MajorRoot$	$F\sharp$	G	$G\sharp$	A	$A\sharp$	B
$Chord$	(6,10,1)	(7,11,2)	(8,0,3)	(9,1,4)	(10,2,5)	(11,3,6)
$MinorRoot$	c	$c\sharp$	d	$d\sharp$	e	f
$Chord$	(0,3,7)	(1,4,8)	(2,5,9)	(3,6,10)	(4,7,11)	(5,8,0)
$MinorRoot$	$f\sharp$	g	$g\sharp$	a	$a\sharp$	b
$Chord$	(6,9,1)	(7,10,2)	(8,11,3)	(9,0,4)	(10,1,5)	(11,2,6)

To avoid confusion with the mathematical notation, we will write our chords as either an ordered triple like we see above, or as the letter in the table with a subscript "major" or "minor". For example, the reader may see either C_{major} or $(0, 4, 7)$ depending on which is more convenient to highlight the mathematical properties at hand.

3. Background: Groups

We will now introduce some useful aspects of group theory which are utilized throughout the article. Mathematicians familiar with elementary group theory may wish to skip this section. We hope that those new to the area will find this section sufficient for the purposes of this article. However, we also hope that such readers will become interested in the topic of group theory, in which case they may want to explore some undergraduate texts on abstract algebra, such as [5] or [6].

A group is a mathematical object often used to understand the symmetry of an object. It can roughly be thought of as a number system with just a single operation and its inverse operation, such as addition and subtraction or multiplication and division. More precisely, a *group* is a set G, together with an operation, say $\star$, that obeys the following axioms:

- *closure*: for any $x, y \in G$, the product $x \star y$ is also in G
- *associativity*: $(x \star y) \star z = x \star (y \star z)$ for any $x, y, z \in G$
- there exists an *identity* element: that is, an element $e \in G$ such that $e \star x = x = x \star e$ for all $x \in G$
- each element has an *inverse*: that is, for each $x \in G$, there is an element $x^{-1} \in G$ satisfying $x \star x^{-1} = e = x^{-1} \star x$.

So, a group is a set that maintains a particular kind of algebraic structure under a specialized operation. In practice, the notation for the operation $\star$ is often compressed and instead written the way one would write ordinary multiplication. Notice that we had no explicit definition of what the elements of a group must look like, and, in fact, do not really care about what these elements look like; as long as they obey the *axioms*, the set they belong to is a group. This provides a kind of useful abstraction that can be used in a variety of situations.

3.1. **Examples of Groups.**

One of the most basic types of groups is called a *cyclic* group, which is a group that can be generated by a single element, a. That is, a group G is cyclic if there is some $a \in G$ such that $G = \langle a \rangle := \{a^i : i \in \mathbb{Z}\}$.

For a finite group, what this definition means is that if we take some element $a \in G$ and perform the binary operation on i iterations of a, written a^i, then at some point we get back to the identity element. We call k the *order* of a if k is the first positive integer such that $a^k = e$. If these iterations cycle through all elements of the group G (that is, if the size of G and the order of a coincide), then G is a cyclic group generated by a. For example $\mathbb{Z}_{12} = \{0, 1, 2, ..., 11\}$, which we used to describe the 12 semitones in the previous section, is a group under addition modulo 12, and as such, is cyclic, generated by the element 1. More generally, the standard example of a cyclic group is $\mathbb{Z}_n$, the set $\{0, 1, 2..., n-1\}$ under addition modulo n.

We will also be particularly interested in a specific type of group called a permutation group. Given a set $\mathcal{A}$, the *symmetric group* $\mathrm{Sym}(\mathcal{A})$ on $\mathcal{A}$ is the set of all permutations on $\mathcal{A}$ (that is, bijective functions $\mathcal{A} \to \mathcal{A}$), under the operation of function composition. In particular, when $\mathcal{A} = \{1, ..., n\}$, we write S_n for $\mathrm{Sym}(\mathcal{A})$. More generally, a *permutation group* on a set $\mathcal{A}$ is any subset of $\mathrm{Sym}(\mathcal{A})$ which is also a group under the same operation of function composition. (In general, we call such a subset a *subgroup*.)

One of the first groups usually encountered in a course on abstract algebra or group theory, and which will be central in our discussion of music, is a special type of permutation group called the *dihedral group of size* n, which we denote by $D_{\frac{n}{2}}$. This group is a set of n permutations that correspond directly to the symmetries on a regular polygon with $\frac{n}{2}$ vertices. For example, D_4 represents the set of symmetries of a square. We may explicitly write the group $D_{\frac{n}{2}}$ as the set generated by an element a of order two and an element b of order $\frac{n}{2}$ such that $aba = b^{-1}$. That is, letting e denote the identity, we may write

$$D_{\frac{n}{2}} = \langle a, b | a^2 = e = b^{\frac{n}{2}}; aba = b^{-1} \rangle. \tag{3.1}$$

What this really means in terms of a shape is that the b permutation is a solid rotation of the $n/2$-gon and the a permutation is a flip about a line of symmetry. By combining these two permutations with multiplication, we can form any possible rotation or flip about a line of symmetry on the $n/2$-gon. It makes sense, then, why there are n permutations in the dihedral group. The b permutation generates $\frac{n}{2}$ unique permutations and when multiplied on the right by a, we double the number of unique permutations, giving us a total of n.

To write down a specific permutation, we will often use array notation. To illustrate this notation, let p denote the permutation on the set $\{1, 2, 3, 4\}$ realized in usual function notation by $p(1) = 2$, $p(2) = 4$, $p(3) = 3$, and $p(4) = 1$. Then, we may compact this into the array: $p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$.

3.2. **Isomorphisms and Direct Products.**

To make abstraction useful in group theory, like in many branches of mathematics, we are often interested in the concept of "sameness". What makes two groups "the same", from a group theory point of view? Abstractly, two groups are considered "the same" if they are *isomorphic*, that is, if there is a particular function called an *isomorphism* between them. More concretely, an isomorphism between two groups G and H is a bijection $\phi\colon G \to H$ such that for all $a, b \in G$, $\phi(ab) = \phi(a)\phi(b)$, where the multiplication on the left-hand side is done under the operation on G and that on the the right-hand side is done under the operation on H.

The concept of isomorphism allows group theorists to organize strange groups into familiar categories, e.g. the statements: a group is dihedral, or, a group is cyclic; mean that a given group is isomorphic to some group within the dihedral or cyclic family, respectively. In particular, it is well-known that if n is a natural number, every cyclic group of size n is isomorphic to $\mathbb{Z}_n$, and that if $\mathcal{A}$ is a set of size n, then $\mathrm{Sym}(\mathcal{A})$ is isomorphic to S_n.

The last notion we introduce from elementary group theory is that of direct products, which describe a way to create a group from two or more smaller groups. Suppose G and H are two groups. Then, as sets, we can create their cartesian product, which is the set $G \times H := \{(g, h) | g \in G, h \in H\}$ of ordered pairs. This set can then be considered as a group, called the *external direct product*, with the operation of component-wise multiplication. That is, if (a, b) and (c, d) are two pairs in $G \times H$, we define $(a, b)(c, d) = (ac, bd)$.

We can convince ourselves that this product is a group since the component-wise multiplication of the pairs forces each individual operation to be done inside a group, i.e. the first component product of the pair in a multiplication is done within G and the second in H.

You may ask: what if the two groups G and H, rather than being abstract groups, are considered as subgroups of some known group, K? If the intersection is trivial, that is, if $H \cap G = \{e\}$, and if every element of H commutes with every element of G, then the set $\{gh : g \in G; h \in H\}$ forms a subgroup of K called the *internal direct product*.

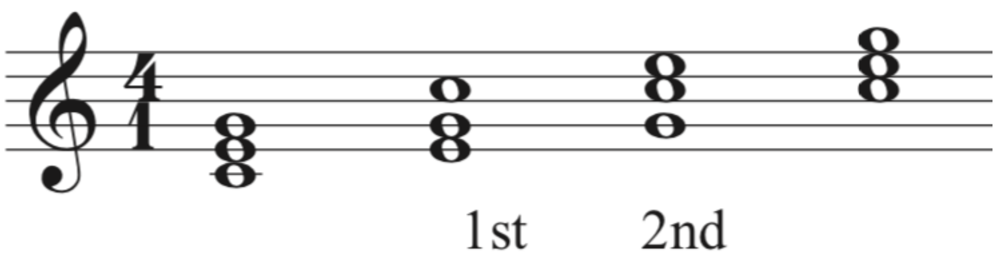
With the internal direct product, the elements of the two groups interact directly as elements of the larger group K, rather than as formal ordered pairs under component-wise multiplication. However, it is well-known that every external direct product is isomorphic to an internal direct product. For this reason, we may use the notation $G \times H$ for either an external or internal direct product.

4. Chord Inversions

We begin our illustration of the appearance of permutation groups in music with a relatively simple example using chord inversions.

Let us begin this examination with a C major chord. As we have seen above, C_{major} can be viewed as the triple $(0, 4, 7)$ of integers modulo 12. Musically, there are two inversions of this chord from its original form. The first is: $(0, 4, 7) \longmapsto (4, 7, 0)$ and the second: $(0, 4, 7) \longmapsto (7, 0, 4)$. We may also choose not to invert the chord, yielding $(0, 4, 7) \longmapsto (0, 4, 7)$. It is important to note that an inverted C major chord is tonally exactly the same as a regular C major chord and so, it is still identified as such.

Visually, this can be represented on a musical staff:



It becomes more convenient to write the above inversions as permutations:

$$(0, 4, 7) \longmapsto (0, 4, 7) = \begin{pmatrix} 0 & 4 & 7 \\ 0 & 4 & 7 \end{pmatrix}$$
$$(0, 4, 7) \longmapsto (4, 7, 0) = \begin{pmatrix} 0 & 4 & 7 \\ 4 & 7 & 0 \end{pmatrix}$$
$$(0, 4, 7) \longmapsto (7, 0, 4) = \begin{pmatrix} 0 & 4 & 7 \\ 7 & 0 & 4 \end{pmatrix}.$$

We see that the first row of the above matrices represents the original chord and the second row represents the inverted chord.

To describe a more general case, let $x_1, x_2, x_3 \in \mathbb{Z}_{12}$ be notes in a chord. We assume these notes form a major or minor triple. For convenience, let $\mathfrak{C}$ denote the set of major and minor triple chords and $\mathfrak{N}$ denote the set of all possible inversions of all major and minor chords. We note that $|\mathfrak{C}| = 24$ and $|\mathfrak{N}| = 3 \cdot 24 = 72$.

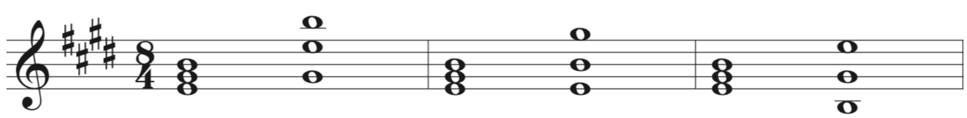
To make the permutations describing inversions precise, define $\rho_n : \mathfrak{N} \to \mathfrak{N}$, for $n = 1, 2, 3$ as:

$$\rho_1 := \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix}, \quad \rho_2 := \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \end{pmatrix}, \quad \text{and} \quad \rho_3 := \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \end{pmatrix}.$$

Notice that we can view each ρ_n as a member of $\mathrm{Sym}(\mathfrak{N})$. Alternatively, given a fixed chord $(x_1, x_2, x_3) \in \mathfrak{C}$, we may view ρ_n as a member of $\mathrm{Sym}(\{x_1, x_2, x_3\})$. With this identification, we may view the set $\mathcal{I}_3 := \{\rho_1, \rho_2, \rho_3\}$ as a subgroup of S_3 by realizing ρ_1 as the identity, noticing that ρ_3 and ρ_2 are inverses of one-another, and that the set is closed under composition. We can further see that $\rho_2^2 = \rho_3$ and $\rho_2^3 = \rho_1$, so that in fact ρ_2 generates the group $\mathcal{I}_3$, which is therefore cyclic of size three. Together, this yields the following:

Lemma 1. The group $\mathcal{I}_3 = \{\rho_1, \rho_2, \rho_3\}$ of chord inversions is isomorphic to $\mathbb{Z}_3$.

4.1. **The actions of Odd Permutations.** The actions of the odd permutations are of some mathematical interest for us. Take for example the $(12) := \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & x_3 \end{pmatrix}$, $(23) := \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_3 & x_2 \end{pmatrix}$, and $(13) := \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_2 & x_1 \end{pmatrix}$ actions on an E major chord. We represent this here with the E major written on a musical staff.



(12) action (23) action (13) action

The actions are permuting $(4, 8, 11)$ thus:

$$\begin{aligned} (4, 8, 11) &\to (8, 4, 11), \\ (4, 8, 11) &\to (4, 11, 8), \\ (4, 8, 11) &\to (11, 8, 4), \end{aligned}$$

respectively. To justify the image further, note that the above permutations are essentially taking two of the 'circles' on the staff and switching them, while leaving the middle in place. However, because of the nature of octaves, this causes somewhat more drastic gaps between the notes. For example, with the first chord action depicted, the 8 becomes the lowest note of the chord, but to get to the next 4 above, we need to traverse 6 semitones. But, because of how the chord triples work, we need to 'shove' the 11 up an octave to preserve the pitch order (i.e. the right-most note on the triple is supposed to be the lowest in pitch and the notes must ascend increasingly with the third note higher than the second, and the second higher than the first).

The justification for the rest of the inversions is the same, though we must note that for the (13) action, we shifted the chord down an octave to make it more visually consistent with the other two (this is valid because of tonal equivalence). This type of chord would certainly show up in musical literature, but it is not precisely a music theoretic construction proper (furthermore, it is not playable on one hand).

5. The $\langle P, L, R\rangle$ Group

Our next illustration of the presence of group theory in music involves the so-called P, L, R *functions*. These are music-theoretic constructions designed by Hugo Riemann in the 1800's to describe how major and minor chords relate to each other. The P, L, R functions are the central topic in the branch of music theory known as Neo-Riemannian theory and have been studied in great detail both mathematically and musically. Here, we are interested in the absolute essentials of these functions, but the curious reader might find [2], [4], [7] interesting for further reading. In addition, [3] provides a nice history of the subject of Neo-Riemannian theory.

The P, L, and R functions are maps from $\mathfrak{C}$ to itself. The P stands for "parallel" and corresponds to the function which maps a major triple to its parallel minor triple and maps a minor triple to its parallel major triple.

The L stands for "leading tone exchange" and corresponds to the function that maps a major chord to a minor chord defined as having a fifth that is one semitone below the root of the major triple and vice versa.

Finally, the R stands for "relative" and corresponds to the function which maps a major triple to a minor triple whose root is two semitones above the fifth of the major triad. Likewise, the R function maps a minor triple to its relative major triple whose fifth is two semitones below the root of the minor triple.

We will use an example to illustrate how these functions work, taking a C_{major} chord as the argument:

$$P((0,4,7)) = (0,3,7),$$

$$L((0,4,7)) = (4,7,11),$$

$$R((0,4,7)) = (9,0,4),$$

and vise versa. It is known that the set $\langle P, L, R\rangle$, of functions on $\mathfrak{C}$ generated by P, L, and R, is a subgroup of $\mathrm{Sym}(\mathfrak{C})$. We see that under function composition (written here as multiplication), $PP = LL = RR$ are the identity of this group, call it e, which shows that each of P, L, and R is its own inverse. Through a counter example, we see that $\langle P, L, R\rangle$ is non-abelian (that is, a non-commutative group):

$$PR((0, 4, 7)) = (9, 1, 4),$$
$$RP((0, 4, 7)) = (3, 7, 10),$$

by direct computation.

The following result is proved in [4] and shows that $\langle P, L, R\rangle$ behaves like one of our well-understood groups. For convenience, we record the result of [4] here:

Lemma 2. ([4]) The $\langle P, L, R\rangle$ group is isomorphic to the dihedral group D_{12} of order 24.

5.1. **On the Subgroup Structure of $\langle P, L, R\rangle$.** More specifically, in [4], it is shown that $P = R(LR)^3$ and $\langle P, L, R\rangle$ can be represented as in (3.1), with $n = 24$, $a = L$ and $b = LR$.

Notice that as a result of [4], we know $\langle L, R\rangle$ is the full group $\langle P, L, R\rangle$. Further, in [2] and [7], it is shown that the subgroup $\langle P, L\rangle$ generated by P and L is dihedral of size 6 (specifically, in [2], this group is represented as in (3.1), with $n = 6$, $a = L$ and $b = LP = (LR)^4$). A natural next question might then be: what can we say about the group $\langle P, R\rangle$?

Let $\langle P, R\rangle$ denote the subgroup of $\langle P, L, R\rangle$ generated by the functions P and R.

Since $P^2 = R^2 = e$, we know $\langle P, R\rangle$ contains the identity and that only alternating sequences of products are distinct. From this we have that

$$\langle P, R\rangle = \{P, PR, R(PR), (PR)^2, R(PR)^2, (PR)^3, R(PR)^3, (PR)^4\}.$$

Further, $(PR)^4$ is the identity function, which we illustrate with the following example:

Consider the behavior of $(PR)^4$ on F_{major}. Note that

$$(PR)^4(F_{major}) = (PR)^3(D_{major}),$$

since the fifth of the F_{major} chord is 0, R gives us the minor chord whose root is two semitones above the fifth, which is 2 in this case, and because P maps minor to major via the third of the minor chord. Similarly,

$$(PR)^3(D_{major}) = (PR)^2(B_{major}) = (PR)(G\sharp_{major}).$$

And finally,

$$(PR)(G\sharp_{major}) = F_{major}.$$

This shows that $(PR)^4$ behaves as the identity on F_{major}, and we encourage the reader to verify with similar calculations that this in fact holds for all chords $Y \in \mathfrak{C}$.

We therefore see that $\langle P, R\rangle$ has size 8, and in fact we prove the following:

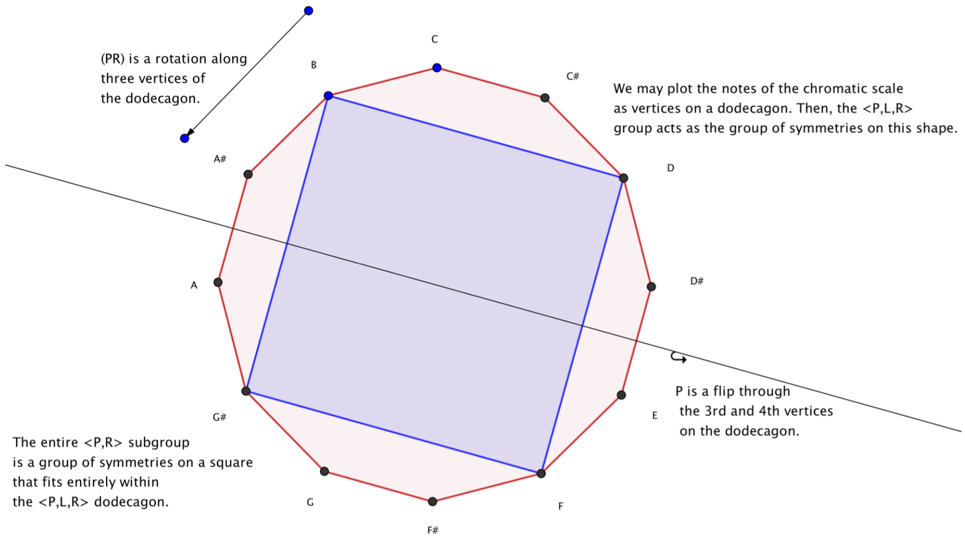
Proposition 1. The subgroup $\langle P, R\rangle$ is isomorphic to the dihedral group D_4 of size eight.

Proof. Since we know $\langle P, R\rangle$ has size 8, we must show that it can be realized in the form $\langle a, b | a^2 = e = b^4; aba = b^{-1}\rangle$. To do this, let $a = P$ and $b = PR$. We saw above that $P^2 = e$ and $(PR)^4 = e$ (and that 2, respectively 4, is the smallest such integer). Further, note that $aba^{-1} = P(PR)P = RP = (PR)^{-1} = b^{-1}$, which proves our claim. □

We can therefore view $\langle P, R\rangle$ as the group of symmetries of a square, and $\langle P, L, R\rangle$ as the group of symmetries of a dodecagon with vertices labeled from $\mathbb{Z}_{12}$. Our goal is to reconcile how $\langle P, R\rangle$ sits inside $\langle P, L, R\rangle$ with the visual of a square embedded in a dodecagon.

As a dihedral group, we have represented $\langle P, R\rangle$ as generated by P and PR. As symmetries of a square (which we may view as sitting within a dodecagon sharing 4 vertices), the P and PR functions act like a flip over a line of symmetry and a rotation of 90 degrees, respectively. But recall that P can also be realized as a flip over a line of symmetry of the dodecagon, and that either LR or its inverse RL can be realized as the generating rotation of the dodecagon. Further, note that $PR = R(LR)^3R = (RL)^3$. That is, the element we have chosen as our generating rotation of the square is exactly the rotation that occurs by applying the generating rotation of the dodecagon 3 times.

This geometric interpretation can be seen below:



But, you may ask, what does this geometric interpretation mean musically? Certainly our geometric interpretation has P and PR operating on the 4 vertices of a square. However, the P and PR functions operate on major and minor triples as opposed to chords with four notes. To understand what the vertices represent, then, we need to instead discuss the orbit of a chord under the action of $\langle P, R\rangle$. That is, we will study the set of chords obtained from a given chord by applying all elements of $\langle P, R\rangle$. The orbit of a C chord, for example, is $\mathcal{O}_{\langle P,R\rangle}(C_{major}) = \{C, c, D\sharp, d\sharp, F\sharp, f\sharp, A, a\}$, which is illustrated in [7].

We can use such an orbit to define an *octatonic system*. As discussed in [7], the octatonic systems are sets of notes based upon the notes required to build the chords in the orbit of a particular major or minor triple acted upon by $\langle P, R\rangle$. This is the set of unique building blocks requisite to construct any chord in the orbit under $\langle P, R\rangle$.

We will continue using the C major chord for our analysis. As given in [7], the octatonic system for C major is $\mathrm{Oct}[\mathcal{O}_{\langle P,R\rangle}(C_{major})] = \{0, 1, 3, 4, 6, 7, 9, 10\}$. Conveniently, when we take the complement, $\{2, 5, 8, 11\}$, of $\mathrm{Oct}[\mathcal{O}_{\langle P,R\rangle}(C_{major})]$ in $\mathbb{Z}_{12}$ to be a subset of vertices inside the dodecagon with labeled vertices from $\mathbb{Z}_{12}$, the resulting image forms the square sitting within the dodecagon that we saw above. We remark that the P function, when applied to the C_{major} chord $(0, 4, 7)$, has the effect of switching 3 and 4, which is why we have represented it in the picture above as flipping the third and fourth vertices. Further, the PR function applied to C_{major} maps $(0, 4, 7)$ to $(9, 1, 4)$, which is indeed obtained by the same rotation of the dodecagon past 3 vertices that rotates our square.

5.2. **A Miscellaneous Result.** We briefly digress to examine one interesting consequence of systems similar to the octatonic one discussed above. In [7], the *hexatonic systems* are derived from the subgroup $\langle L, P\rangle \leq \langle P, L, R\rangle$ in a manner analogous to the octatonic systems. The resulting set is given as $\mathrm{Hex}[\mathcal{O}_{\langle L,P\rangle}(C_{major})] = \{0, 3, 4, 7, 8, 11\}$. These functions Hex and Oct may be generalized in the following way: given some major or minor chord $X := (x_1, x_2, x_3)$, define

$$\mathrm{Hex}[\mathcal{O}_{\langle L,P\rangle}(X)] = \{x_1, x_1 + 3, x_1 + 4, x_1 + 7, x_1 + 8, x_1 + 11\}, \quad \text{and}$$

$$\mathrm{Oct}[\mathcal{O}_{\langle P,R\rangle}(X)] = \{x_1, x_1+1, x_1+3, x_1+4, x_1+6, x_1+7, x_1+9, x_1+10\}.$$

Then, we may take the intersection of these sets:

$$\mathrm{Hex}[\mathcal{O}_{\langle L,P\rangle}(X)] \cap \mathrm{Oct}[\mathcal{O}_{\langle P,R\rangle}(X)] = \{x_1, x_1 + 3, x_1 + 4, x_1 + 7\},$$

which is, beautifully, the set of all notes necessary to make a major or minor chord with root x_1.

6. Combining These Chord Actions

In this section, we wish to describe how the previous two examples interact. It is clear that both the $\langle P, L, R\rangle$ and $\mathcal{I}_3$ groups operate upon major and minor triples. Along the same lines, it is musically valid to play, for example, major C, minor c, then inverted minor c, chords in sequence. It is also musically valid to play inversions of relative minor chords or to change a chord from a major to a minor while maintaining some kind of inversion. Mathematically, this implies that $\langle P, L, R\rangle$ and $\mathcal{I}_3$ groups may operate in conjunction with each other.

However, as we have seen before, the $\langle P, L, R\rangle$ functions only operate on the chord type, while the $\mathcal{I}_3$ functions operate on the order of the tones within a chord, meaning that $\langle P, L, R\rangle$ may be viewed as a subgroup of $\mathrm{Sym}(\mathfrak{C})$, whereas $\mathcal{I}_3$ is a subgroup of $\mathrm{Sym}(\mathfrak{N})$. Notice that a chord having undergone an inversion does not change its identification. That is to say, an inverted chord is still tonally the same as it was before the inversion.

It makes sense to reconcile this issue by extending the action of the P, L, and R functions in a natural way to inverted major and minor chords. Namely, if $\sigma \in \langle P, L, R\rangle$ and σ is applied to an inversion of the chord c, we simply take the result to be the corresponding inversion of $\sigma(c)$. That is to say, if $(x, y, z) \in \mathfrak{C}$ and $\rho_n \in \mathcal{I}_3$, we define $\sigma(\rho_n(x, y, z)) := \rho_n(\sigma(x, y, z))$. This then allows us to consider $\langle P, L, R\rangle$ as a subgroup of $\mathrm{Sym}(\mathfrak{N})$. Further, notice that every element of $\langle P, L, R\rangle$ commutes with every element of $\mathcal{I}_3$, by our construction.

Viewing these two groups as subgroups of the symmetric group $\mathrm{Sym}(\mathfrak{N})$ on the set of all inversions of major and minor chords, we may define a subset $\Phi \subseteq \mathrm{Sym}(\mathfrak{N})$ as the set $\Phi = \langle P, L, R\rangle \cdot \mathcal{I}_3 = \{\sigma\rho | \sigma \in \langle P, L, R\rangle; \rho \in \mathcal{I}_3\}$.

Proposition 2. The set Φ forms a subgroup of $\mathrm{Sym}(\mathfrak{N})$. Moreover, Φ is an internal direct product of $\langle P, L, R\rangle$ and $\mathcal{I}_3$.

Proof. By observing the way $\langle P, L, R\rangle$ and $\mathcal{I}_3$ behave on $\mathfrak{N}$, we see that $\langle P, L, R\rangle \cap \mathcal{I}_3 = \{e = \rho_1\}$, yielding that Φ contains the identity and is generated by two subgroups of $\mathrm{Sym}(\mathfrak{N})$ who intersect only at the identity.

To prove that Φ is the stated internal direct product, it therefore suffices to note that $\sigma\rho = \rho\sigma$ for all $\sigma \in \langle P, L, R\rangle$ and all $\rho \in \mathcal{I}_3$ from our discussion above.

Finally, by the consequences of the internal direct product, it must also be that Φ is a subgroup of $Sym(\mathfrak{N})$. $\square$

Since we know that $\mathcal{I}_3$ and $\langle P, L, R\rangle$ are isomorphic to $\mathbb{Z}_3$ and D_{12}, respectively, and that an internal direct product is always isomorphic to an external direct product, we have shown the following:

Corollary 1. The group Φ is isomorphic to the external direct product $D_{12} \times \mathbb{Z}_3$.

Transposition and The Circle of Fifths. In this section, we illustrate another application of permutation groups to music theory, this time involving musical actions called *transpositions.* These actions are studied in great detail in [4], in the context of a group known as the T/I Group. However, here we will highlight an interesting consequence of these functions only.

Let the functions $\mathcal{T}_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ for $n \in \mathbb{Z}_{12}$ be defined by

$$\mathcal{T}_n(x) := x + n \pmod{12}.$$

These functions represent musical transpositions, which simply take any note $x \in \mathbb{Z}_{12}$ and shift it by an amount $n \in \mathbb{Z}_{12}$. Performing one of these operations is like sliding one's finger along a keyboard where the starting key is called x and the distance is n. It is shown in [4] that these functions act point-wise on chords. So, by abuse of notation, we may also consider $\mathcal{T}_n$ as functions $\mathcal{T}_n : \mathbb{Z}_{12}^k \to \mathbb{Z}_{12}^k$ for some $k \in \mathbb{N}$ representing the number of notes in the chord.

We will specifically consider the function $\mathcal{T}_7$, which operates by $\mathcal{T}_7(x) = x + 7 \pmod{12}$, and denote by $\langle \mathcal{T}_7 \rangle$ the group generated by $\mathcal{T}_7$ under function composition (which we will represent as multiplication).

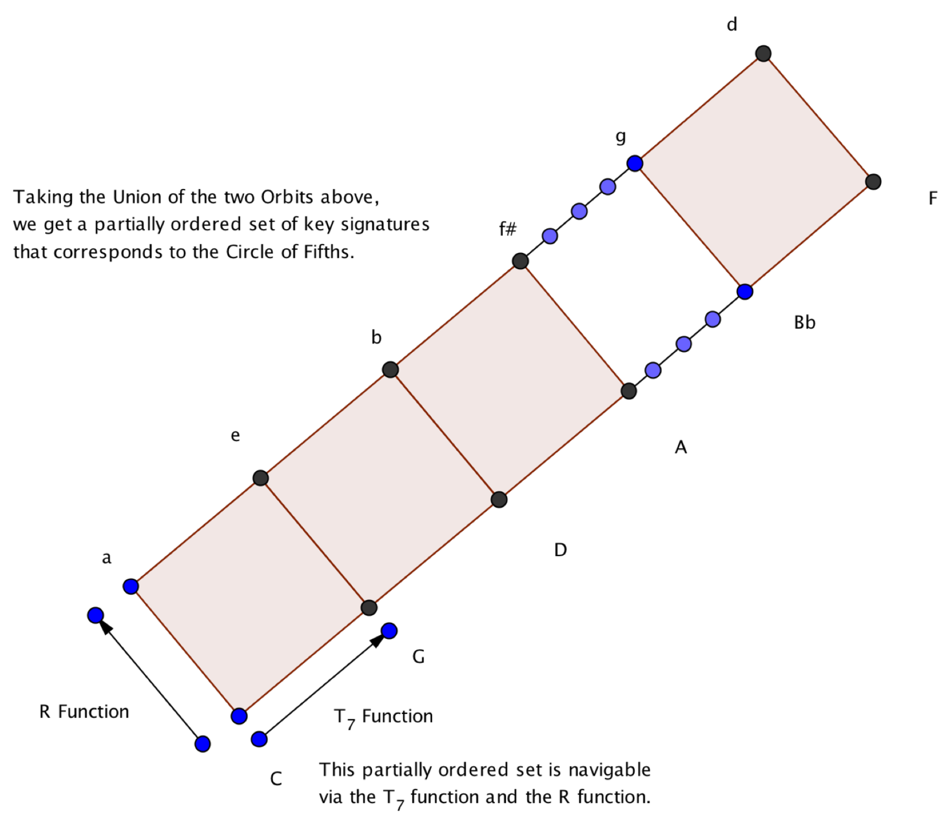
Since 7 is relatively prime to 12, it follows that 7 generates $\mathbb{Z}_{12}$. That is, $n7$, as n ranges over all elements of $\mathbb{Z}$, will cycle through all equivalence classes in $\mathbb{Z}_{12}$. The consequence of this is that $\mathcal{T}_7^n$ will cycle through all $\mathcal{T}_r$ for $r \in \mathbb{Z}_{12}$. Explicitly, we may write:

$$\langle \mathcal{T}_7 \rangle = \{\mathcal{T}_0, \mathcal{T}_7, \mathcal{T}_2, \mathcal{T}_9, \mathcal{T}_4, \mathcal{T}_{11}, \mathcal{T}_6, \mathcal{T}_1, \mathcal{T}_8, \mathcal{T}_3, \mathcal{T}_{10}, \mathcal{T}_5\}.$$

We denote by $\mathcal{O}_{\langle \mathcal{T}_7 \rangle}(C)$ the orbit of the note C (or, depending on context, the C major scale) under the action of $\langle \mathcal{T}_7 \rangle$. That is, $\mathcal{O}_{\langle \mathcal{T}_7 \rangle}(C)$ is the set of notes obtained by applying $\mathcal{T}_7$ repeatedly. Then, $\mathcal{O}_{\langle \mathcal{T}_7 \rangle}(C) = \{C, G, D, A, E, B, F\sharp, D\flat, A\flat, E\flat, B\flat, F\}$. The musically inclined reader will quickly notice that we have generated the major portion of the circle of fifths!

Now, consider the set $R[\mathcal{O}_{\langle \mathcal{T}_7 \rangle}(C)]$, comprised of the set obtained by multiplying each element in the above orbit (identified as a major triple) on the left by R. This is the relative minor portion of the circle of fifths. This yields an interesting visual described in the following proposition and illustrated below.

Proposition 3. The union $\mathcal{O}_{\langle \mathcal{T}_7 \rangle}(C) \cup R[\mathcal{O}_{\langle \mathcal{T}_7 \rangle}(C)]$ forms a commutative diagram that is navigable by combinations of the $\mathcal{T}_7$ and R functions.



7. Some Concluding Remarks

Although we've seen several examples of how permutation groups can be used to describe the structure of musical chords, we've barely scratched the surface of the hidden group theory within music.

One could go on, for example, to use groups to describe the chord progressions of an ensemble, or more simply the motions of the hand of a pianist progressing through a series of chords that make up a song. Using 5-tuples of elements of $\mathbb{Z}_{12}$ to describe the chords a given hand is playing, the change-of-chord could be described by adding another 5-tuple, yielding a group isomorphic to $\mathbb{Z}_{12}^5$ (that is, the direct product of $\mathbb{Z}_{12}$ with itself 5 times) that describes the progression of a single hand from one chord to another. Considering both hands playing at once and the possibility of "crossing over" the hands, we could reasonably use the group $\mathbb{Z}_{12}^5 \times \mathbb{Z}_{12}^5 \times S_2$ to describe the possibilities of chord progressions for the piano player. (We remark that some simplification has to be made in this example to rectify the situation that, at some point a hand might be playing fewer than five notes. To do this, we have to realize that $(0, 0, 4, 4, 7)$, $(0, 4, 4, 4, 7)$, and so on, should all be the same musically.)

Further examples are plentiful, but it is the author's hope that this article has met its goal of the illumination of the crossover between the beautifully abstract study of symmetry that is group theory and the fundamental part of culture that is music, adding a piece atop the existing literature about the hidden mathematical symmetries of music.

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